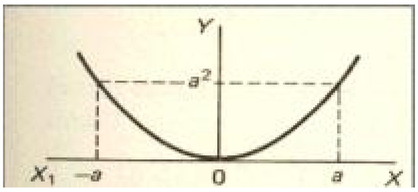
Lecture # 22

Odd And Even Functions

(a) Even functions

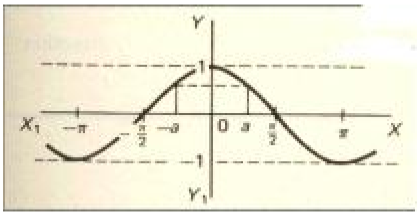
A function f(x) is said to be even if $f(-x) = f(x)$ i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x. The graph of an even function is therefore symmetrical about the y-axis.



$y = f(x) = x^2$ is an even function since

$f(-2) = 4 = f(2)$

$f(-3) = 9 = f(3)$ etc.

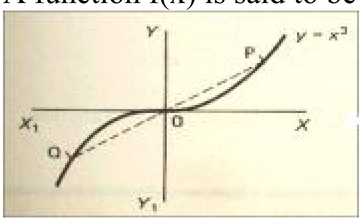


$y = f(x) = \cos x$ is an even function since $\cos(-x) = \cos x$

$f(-a) = \cos a = f(a)$

(b) Odd functions

A function f(x) is said to be odd if $f(-x) = -f(x)$

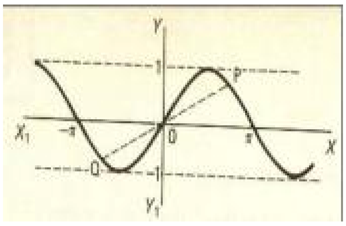


i.e. the function value for a particular negative value of xis numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus symmetrical about the origin.

$y = f(x) = x^3$ is an odd function since

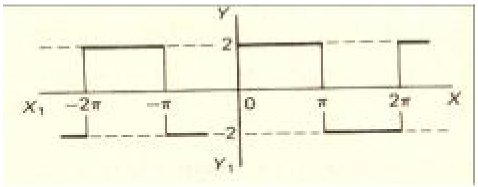
$f(-2) = -8 = -f(2)$

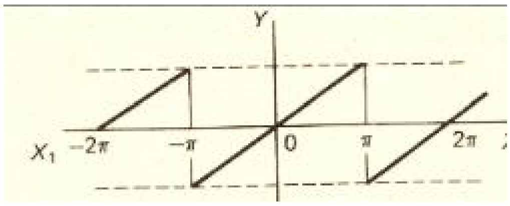
$f(-5) = -125 = -f(5)$ etc.



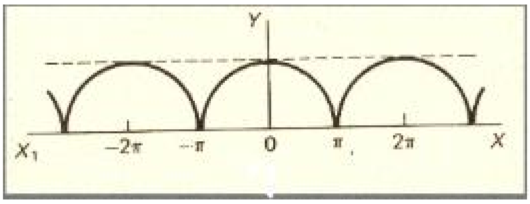
y = f(x) = sin x is an odd function Since
sin (– x) = – sin x
f (– a) = – f(a).

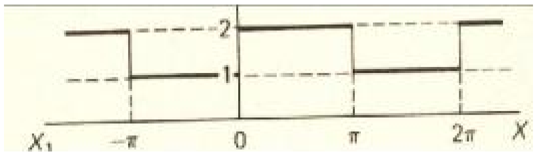
So, for an even function f (– x) = f(x)
symmetrical about the y-axis
for an odd function f(– x) = – f (x) symmetrical about the origin.



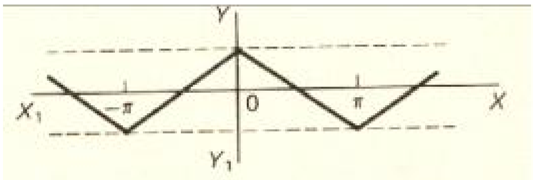


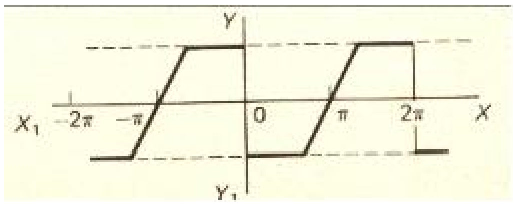
odd. Odd





Even   neither





Even   Odd

<u>Products Of Odd And Even Functions</u>

The rules closely resemble the elementary rules of signs.
(even)×(even)=(even) like (+)×(+)=(+) ; (odd)× (odd) = (even)   (–)×(–)=(+) ;
(odd)× (even) = (odd) (–)×(+)=(–)
The results can easily be proved.

(a) Two even functions
Let F(x) = f(x) g(x) where f(x) and g(x) are even functions.
Then F(– x) = f(– x) g (– x) = f(x) g(x) since f(x) and g(x) are even.
$\therefore$ F(– x) = F(x)
F(x) is even

(b) Two odd functions
Let F(x) = f(x) g(x) where f(x) and g(x) are odd functions.
Then F(– x) = f(– x) g(– x) = {– f (x)} {– g(x)}
since f(x) and g(x) are odd.
= f(x) g(x) = F (x)
$\therefore$ F(– x) = F(x)
F (x) is even
Finally

(c) One odd and one even function
Let F(x) = f(x) g(x) where f(x) is odd and g(x) even.
Then F (– x) = f (– x) g(– x) = – f(x) g(x) = – F (x)
$\therefore$ F (– x) = – F (x)

F (x) is odd

So if f(x) and g(x) are both even, then f(x) g(x) is even and if f(x) and g(x) are both odd, then f(x) g(x) is even but if either f(x) or g(x) is even and the other odd. Then f(x) g(x) is odd.

Example

State whether each of the following products is odd, even, or neither.

1. $x^2 \sin 2x$ odd (E) (O) = (O)
2. $x^3 \cos x$ odd (O) (E) = (O)
3. $\cos 2x \cos 3x$ even (E) (E) = (E)
4. $x \sin nx$ even (O) (O) = (E)
5. $3 \sin x \cos 4x$ odd (O) (E) = (O)
6. $(2x + 3) \sin 4x$ neither (N) (O) = (N)
7. $\sin^2 x \cos 3$ xeven (E) (E) = (E)
8. x^3e^x neither (O) (N) = (N) 9.

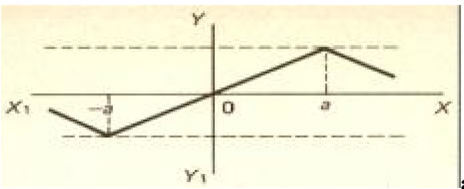
$(x^4 + 4) \sin 2x$ odd (E) (O) = (O)

Two useful facts emerge from odd and even functions.

(a) For an even function

$$\int_{-a}^{a} f(x)dx=2\int_{0}^{a} f(x)\, dx$$

(b) For an odd function



$$\int_{-a}^{a} f(x)\, dx = 0$$

THEOREM 1

If f(x) is defined over the interval $-\pi < x < \pi$ and f(x) is even, then the Fourier series for f(x) contains cosine terms only. Included in this is a_0 which maybe regarded as $a_n \cos nx$ with n = 0.

Proof:

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx \therefore \quad a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx$

(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}$ f(x) cos nx dx.

But f(x) cos nx is the product of two even functions and therefore itself even.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx. \therefore a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx$

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}$

$\int_{-\pi}$ f(x) sin nx dx

Since f(x) sin nx is the product of an even function and an odd function, it is itself odd.

$\pi \therefore\ b_n = \frac{1}{\ }^{\pi}$

$\therefore\ b_n = 0$

$\int_{-\pi}$ f(x) sin nx dx = 0.

Therefore, there are no sine terms in the Fourier series for $f(x)$.